Basic maths recap

We will be using a couple of high-school formulas, so a little recapitulation won't hurt.

- Magnitude of a vector: $|v| = \sqrt{v_x^2 + v_y^2}$
- Scalar multiplication of two vectors: $v \cdot u = v_x \cdot u_x + v_y \cdot u_y$
- Angle between two normalized vectors: $\alpha = \arccos(v \cdot u)$
- Linear function equation: $y = kx + q$, where q is an offset off the Y-axis and k is a tangent of an angle between the line representing the linear function and the X-axis, $\tan(\alpha)$ then equals to $\frac{y-q}{x}$

How to normalize a vector

A normalized vector is a vector that has a magnitude of 1. The aforementioned formula to calculate the angle between vectors works only with normalized vectors and the chances are that we will not get a normalized vector using some real data. To get a normalized vector from a non-normalized one, we can use following logic.

Express the Y-size of a normalized vector like this

$$v_{ny}^2 = |v|^2 - v_{nx}^2$$

And express Y-size using the linear function equation $y = \tan(\alpha)x$, so we get (assume q is 0 which is true if both vectors radiate from the same point)

$$\tan^2(\alpha)v_{nx}^2 = |v|^2 - v_{nx}^2 \ / + v_{nx}^2$$

$$v_{nx}^2(\tan^2(\alpha) + 1) = |v|^2$$

... and finally ...

$$v_{nx} = \sqrt{\frac{|v|^2}{\tan^2(\alpha) + 1}}$$

Now if we set $|v| = 1$ and calculate $\tan(\alpha)$ using the X and Y sizes of the original vector, we get the normalized X-size of the vector.
The Y-size is then calculated like this

$$v_{ny} = \tan(\alpha)v_n x$$

Example:
Normalize a vector v(2,3)

$$\tan\alpha = \frac{3}{2} = 1.5$$

$$v_{nx} = \sqrt{\frac{1}{3.25}} \approx 0.555$$

$$v_{ny} = 1.5 \cdot 0.555 \approx 0.833$$

Check that the resulting vector is normalized

$$|v_n| = \sqrt{0.555^2 + 0.833^2} \approx 1.001$$

Calculate an angle between two vectors using the method described above

- **Example 1**: v(2,3), u(2,7)
  Normalize both vectors
  $v_{nx} = 0.555,\ v_{ny} = 0.833$ and $u_{nx} = 0.275,\ u_{ny} = 0.936$

  Do a scalar multiplication of the normalized vectors
  $v_n \cdot u_n = 0.555 \cdot 0.275 + 0.833 \cdot 0.936 = 0.961$

  Calculate the angle between these vectors
  $\arccos 0.961 = 16.05°$
- **Example 2**: v(2,3), u(9,-1)
  Normalize both vectors
  $v_{nx} = 0.555,\ v_{ny} = 0.833$ and $u_{nx} = 0.994,\ u_{ny} = -0.110$

  Do a scalar multiplication of the normalized vectors
  $v_n \cdot u_n = 0.555 \cdot 0.994 + 0.833 \cdot (-0.110) = 0.460$

  Calculate the angle between these vectors
  $\arccos 0.460 = 62.61°$
- **Example 3**: v(-3,5), u(-6,-2)
  Normalize both vectors
  $v_{nx} = -0.514,\ v_{ny} = 0.857$ and $u_{nx} = -0.949,\ u_{ny} = -0.316$

  Do a scalar multiplication of the normalized vectors
  $v_n \cdot u_n = (-0.514) \cdot (-0.949) + 0.857 \cdot (-0.316) = 0.217$

  Calculate the angle between these vectors
  $\arccos 0.217 = 77.47°$

Notice that the signs of X and Y sizes of normalized vectors in examples 2 and 3 match those of the original vectors.